Quasi-treeable CBERs are treeable via median graphs

Ran Tao

Carnegie Mellon University

Joint with Ruiyuan Chen (University of Michigan), Antoine Poulin (McGill) and Anush Tserunyan (McGill)

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Goal of the talk

Theorem (Main theorem)

A CBER (X, E) is properly wallable if and only if it is treeable.

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Theorem

If a CBER (X, E) admits a locally finite graphing whose components are quasi-trees then it is treeable.

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CBERs

Definition (CBER)

Let X be a Polish space. A *countable Borel equivalence relation (CBER)* E on X is a Borel subset of X^2 such that each E-class is countable.

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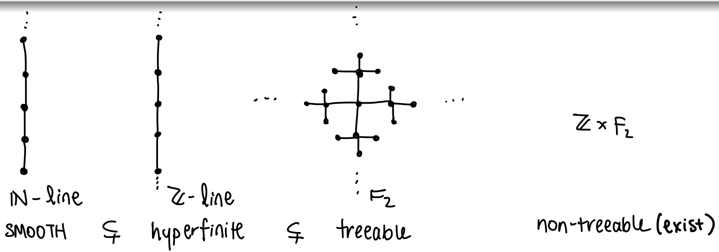
Definition (graphing)

A Borel $G \subset E \subset X^2$ is a *graphing* of E if the connected components of G are precisely the equivalence classes of E.



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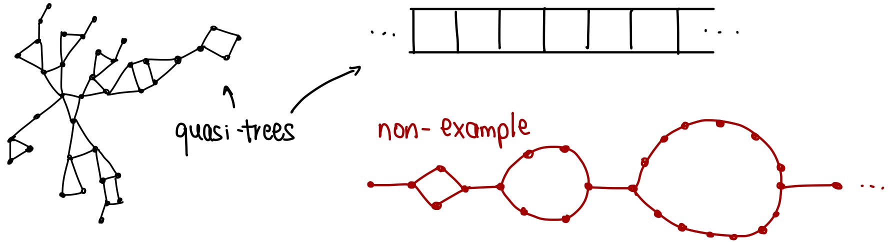
Example.



Motivation

Theorem

The orbit equivalence relation of a free action of a free group is treeable.

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A finitely generated group which has a Cayley graph quasi-isometric to a tree is virtually free.

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Question

Are quasi-treeable CBERs treeable?

Game plan

We have a plan.

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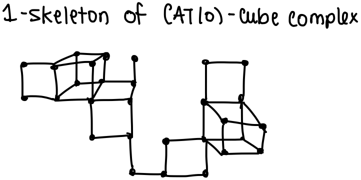
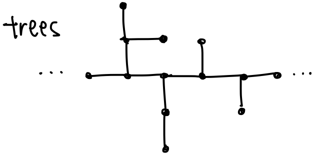
Median graphs

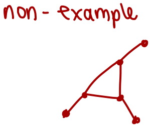
Definition (Median graph)

(X, G) is a *median graph* if it is connected and for any $x, y, z \in X$, $[x, y] \cap [y, z] \cap [x, z] = \{\langle x, y, z \rangle\}$ is a singleton.

Examples.







Some definitions

Let (X, G) be a median graph.

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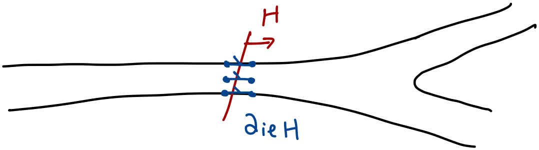
For $H \in \mathcal{H}$, the directed inner edge boundary $\partial_{ie} H \subset G$ is a *hyperplane*.



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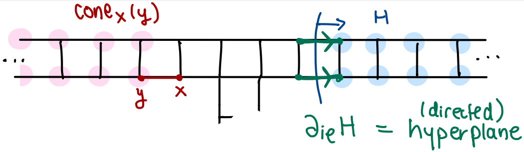
For $x, y \in X$ $\text{cone}_x(y) = \{z \in X : d(x, z) > d(y, z)\} = \{z : (x - y - z)\}$



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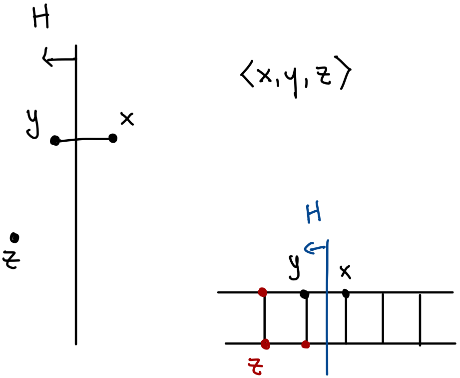
Definition

A set X with a collection $\mathcal{H} \subset 2^X$ is a *wallspace* if for any $x, y \in X$ there are only finitely many $H \in \mathcal{H}$ separating x and y.

Cones

Lemma

Each non-trivial $H \in \mathcal{H}$ is equal to $\mathrm{cone}_x(y)$ for any $(x, y) \in \partial_{ie} H$.



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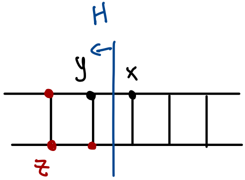
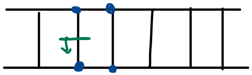
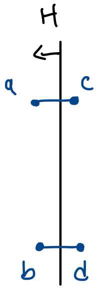
Lemma

Squares generate hyperplanes.







Nested and successor

Definition

Let H, K be half-spaces. Then H is

1. *nested* with K if H is comparable with one of $K, \neg K$ under inclusion,
2. a *successor* of K if $K \subsetneq H$ and there is no half-space strictly in between.

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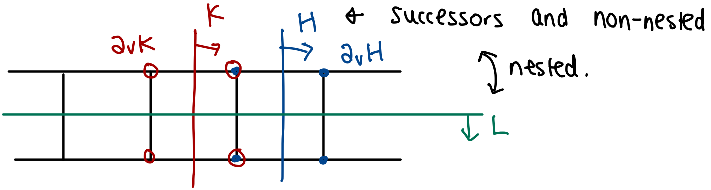
Note: ~~This happens~~ non-nested OR successor if and only if $\partial_v H \cap \partial_v K \neq \emptyset$, or if $\{H, \neg H\} = \{K, \neg K\}$.



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Stone-duality theorem

Theorem (Isbell, Werner)

There is a contravariant equivalence of categories between
$\{(X, G)$ *median, median homomorphisms*$\} \to$
$\{$*pocsets*$^*(P, \leq, \neg, 0)$, *continuous homomorphisms*$\}$.

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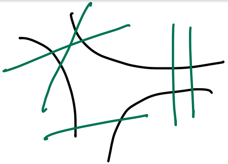
Definition

An *orientation* on $\mathcal{H}$ is an upward-closed subset $U \subset \mathcal{H}$ containing exactly one of $H, \neg H$ for each $H \in \mathcal{H}$.
Let $\mathcal{U}(\mathcal{H})$ denote the collection of orientations on $\mathcal{H}$.
Let $\mathcal{U}^\circ(\mathcal{H}) \subset 2^{\mathcal{H}}$ denote the collection of clopen orientations on $\mathcal{H}$.



From median graph to wallspace

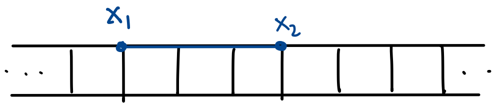
median graph $\longrightarrow$ wallspace

$$(X, G) \longrightarrow \mathcal{H}_{cvx}(X) \subseteq 2^X$$

$$(X, G) \xrightarrow{\simeq} \mathcal{U}^\circ(\mathcal{H}_{cvx}(X)) \quad \text{is an isomorphism}$$

$$x \longmapsto \hat{x} = \{ H \in \mathcal{H}(X) : x \in H \}$$

Surjective :



From wallspace to median graph

wallspace $\mathcal{H}$ $\longrightarrow$ median graph $\mathcal{U}^\circ(\mathcal{H}) \subseteq 2^{\mathcal{H}}$

and $\mathcal{H} \longrightarrow \mathcal{H}_{cvx}(\mathcal{U}^\circ(X))$ is an isomorphism.

$H \longmapsto \{U \in \mathcal{U}^\circ(X) \mid H \in U\}$

surjective :

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Properly wallable

Definition

A set X with a collection $\mathcal{H} \subset 2^X$ is a *wallspace* if for any $x, y \in X$ there are only finitely many $H \in \mathcal{H}$ separating x and y (i.e. *finitely separating*). ($\mathcal{H}$ is a *walling* of X.)

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Definition

A wallspace is *proper* if it satisfies

1. each $\mathcal{H}$-block is finite (i.e., for any x, there are finitely many y with $\forall H \in \mathcal{H}, x \in H \iff y \in H$,
2. for any $H \in \mathcal{H}$, there are only finitely many $K \in \mathcal{H}$ non-nested with H,
3. for any $H \in \mathcal{H}$ there are only finitely many successors $H \subsetneq K \in \mathcal{H}$.

Note: one can define a *Borel (proper) walling*.

Examples of proper wallings

Lemma

If $\mathcal{H}$ is a proper walling, then $\mathcal{U}^{\circ}(\mathcal{H})$ has finite hyperplanes.

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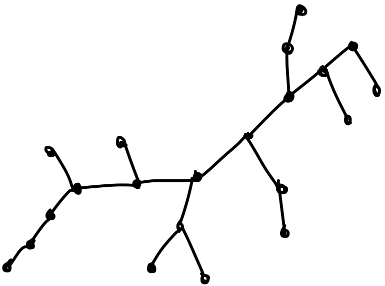
If $\mathcal{H}$ is a proper walling, then $\mathcal{U}^{o}(\mathcal{H})$ has finite hyperplanes.

Examples

Locally finite trees are properly wallable.



Quasi-isometry and properly wallable

Theorem

A CBER (X, E) is properly wallable if and only if it is treeable.

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Proof.

Let $(Y, \mathcal{H})$ be a countable proper wallspace. Consider the median graph $\mathcal{U}^{\circ}(\mathcal{H}) =: (X, G)$ with finite hyperplanes, we construct a subtree $T \subset G$.

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1. We countably colour $\mathcal{H}(X) = \sqcup_n \mathcal{H}_n$, such that $H, \neg H$ receive the same colour, and non-nested half-spaces get different colours.

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2. Construct the equivalence relations K_n as for $x, y \in X$, $xK_ny \iff (\forall k > n, \forall H \in \mathcal{H}_k, x \in H \iff y \in H)$. Then $\cup_n K_n = X^2$.

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3. K_n treed, we tree K_{n+1}. If A, B are K_n-blocks that are in the same K_{n+1}-block, pick an edge.

$H \mapsto \{U \in \mathcal{U}^o(\mathcal{H}) \mid U \ni H\}$

$(Y, \mathcal{H})$ is Borel bireducible with (X, G) by $y \mapsto \hat{y} = \{\}$. □

In case drawing works

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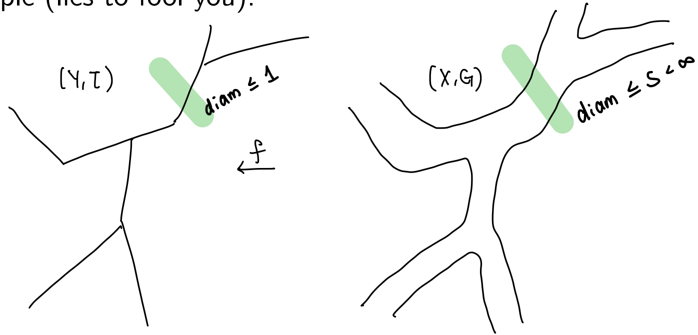
Quasi-isometry and properly wallable

Theorem

If $f : (X, G) \to (Y, T)$ is a quasi-isometry, and (Y, T) is properly wallable with $\mathcal{H}_{diam(\partial \leq R)}$ for $R < \infty$, then (Y, T) is properly wallable.

An example (lies to fool you).



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Thus, quasi-trees are properly wallable.

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Thus, quasi-trees are treeable. □

Summary

~~Add the summary.~~ Here it is.

Last slide

The end.